MATH 3060 Tutorial 1

Chan Ki Fung

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1. Determine whether the followings are true or not.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges absolutely.

   (c) If f, g are functions integrable on $[0,1]$, then so is the product fg.

   (d) $\sum_{n=1}^{\infty}(x^{n-1} - x^n)$ converges pointwise on $[0,1]$.

   (e) $\sum_{n=1}^{\infty}(x^{n-1} - x^n)$ converges uniformly on $[0,1]$.

   (f) If $\sum_n s_n(x)$ converges uniformly and absolutely, and each $s_n(x)$ is differentiable, then $\sum_n s_n'(x)$ converges uniformly and absolutely.

   (g) If $\sum_n s_n'(x)$ converges uniformly and absolutely, then $\sum_n s_n(x)$ converges uniformly and absolutely. Moreover

   $$\sum_n s_n'(x) = (\sum_n s_n(x))'.$$

   (h) If $\sum_n s_n(x)$ converges uniformly and absolutely, and each $|s_n(x)|$ is integrable on $[0,1]$, then $\sum_n \int_0^x s_n(t)dt$ converges uniformly and absolutely. Moreover,

   $$\sum_n \int_0^x s_n(t)dt = \int_0^x \sum_n s_n(t)dt.$$

   (i) If $s(x) = \sum s_n(x)$ converges uniformly on $[0,1]$, and each $s_n(x)$ is continuous on $[0,1]$, then $s(x)$ is uniformly continuous on $[0,1]$.

   (j) If $f(x)$ is a 2π periodic function, then $f(x)$ and $f(x-c)$ have the same Fourier series for any real constant c.

   ANS: T,F,T,T,F,F,F,T,T,F

2. (M-test) If $\sum_n M_n < \infty$ and $|s_n(x)| \leq M_n$ for all n and x. Then $\sum_n s_n(x)$ converges uniformly and absolutely.

3. Find the Fourier series of the function $f(x) = |x|, x \in [-\pi, \pi]$

4. Let f be a 2π periodic odd function, with $f(x) = x(\pi - x)$ for $x \in [0, \pi]$. Find the Fourier seriese of f.

5. Let f be a 2π-periodic function which is infinitely many times differentiable on $\mathbb{R}$. Show that its Fourier coefficients are of order $o(1/n^k)$ for any $k \geq 1$.
   Hint: What is the relation between $c_n(f)$ and $c_n(f')$?

6. Let f, g be continuous 2π periodic, define h by

$$h(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y)dy.$$

   Show that $c_n(h) = c_n(f)c_n(g)$.
   Hint: You may write $e^{-inx} = e^{-in(x-y)}e^{-iny}$ to apply Fubini theorem.